Chapters 2
Differential Topology
Guillemin & Pollack

Name: Jiajie (Jerry) Luo

In the first chapter, we discussed what a differentiable manifold is and some of the properties they have. We notice however that there are some objects, such as the closed unit disk, the cylinder $S^1 \times [0,1]$, and the upper half space $H^k = \{(x_1, \cdots, x_k) \in \mathbb{R}^k : x_k \geq 0\}$, that are "like" manifolds but aren't under our definition. In order to remedy this, we introduce a wider scope of objects: manifolds with boundary. We use the space H^k as a model space for k-dimensional manifolds with boundary. That is, $X \subset \mathbb{R}^n$ is a k-manifold with boundary if every $x \in X$ has a neighborhood diffeomorphic to an open set in H^k in its restricted topology. We call denote the boundary points of X, ∂X, as the points in X that have neighborhoods diffeomorphic to a neighborhood of H^k that has boundary points. We denote $Int(X) = X \setminus \partial X$. We note that this concept of boundary is NOT the same as the point set topology definition (consider S^k embedded in $\mathbb{R}^{k+1}$). Here, we note that ∂X is also a manifold (without boundary), of dimension $k-1$.

Unlike normal manifolds, taking the product of two manifolds with boundaries does not give another manifold with boundary in general. To see where this fails, consider $[0,1] \times [0,1]$. However, given a manifold without boundary X and a manifold with boundary Y, we note that $X \times Y$ is a manifold with boundary, with $\partial(X \times Y) = X \times \partial Y$, and $\dim(X \times Y) = \dim(X) + \dim(Y)$.

Now, given a k-manifold with boundary X, it makes sense for us to look at the tangent space of X at a point x, and in extension, the derivative of a map g on X. If $x \in Int(X)$, we see that the same definition for $T_x(X)$ for manifolds without boundaries still works. But what if $x \in \partial X$? It turns out, the same definition for this works as well, by a consequence of g being a smooth map, which also extends to local charts for points in ∂X (indeed, local charts are smooth). In this case, $T_x(X)$ is a k-dimensional vector space, even when $x \in \partial X$. As before, if $x \in \partial X$, we have $T_x(\partial X)$ is a $k-1$ dimensional manifold, and also a subspace of $T_x(X)$. We actually have all the nice properties of derivative maps, such as the chain rule. From here on out, for a map f on X, we refer to $f|_{\partial X}$ as ∂f. We see that with this notation, we also have that the derivative of ∂f at x is simply the restriction of df_x to $T_x(\partial f)$.

Some of the ideas that we introduced in the first chapter are almost the same, except with a few slight changes. One of these is Sard's theorem. In the manifold with boundary case, we have that given $f : X \to Y$ where X has boundary and Y is without boundary, almost every $y \in Y$ is a regular value for both f and ∂f. This makes sense, as f restricted on ∂X gives a smooth map from a manifold without boundary.

Before we go further, we will go into a digression on compact connected 1-dimensional manifolds with boundary that will become useful later. It turns out that every compact connected 1-dimensional manifold with boundary is diffeomorphic to either $[0,1]$ or S^1. Intuitive, this kind of makes sense, because since we have the compact and connected condition, we can either "run in circles" or "start somewhere and stop." An immediate correlary of this is that every compact 1-dimensional manifold with boundary has an even number of points in its boundary. This makes sense, as each path-component is either diffeomorphic to S^1 or $[0,1]$; in the first case, there are no boundary points, and in the second, there are 2. Although this seems like a rather week result, it gives us quite a bit of mileage. In particular, we can use the corollary to show that given X, a compact manifold with boundary, then there is no smooth maps $g : X \to \partial X$ where $\partial g : \partial X \to \partial X$ is the identity. We can go even further in showing the Brouwer fixed-point theorem, which states that every function on the n-dimensional closed ball to itself must have fixed points.

Another concept that can be slightly tweaked is the idea of transversality, and how we can use it to determine whether the pull back of a manifold is a manifold. We in fact can use this same idea for manifolds with boundary. We recall in the first chapter that given $f : X \to Y$ with $Z \subset Y$, we have that $f^{-1}(Z)$ is a submanifold of X given that f is transversal to Y. Here, the definition of transversality carries over. It turns out that in the case for manifolds with boundary, a similar condition arises. However, this time, we need the extra assumption that ∂f is also transversal to Z. That is, given $f : X \to Y$ where X is manifold with boundary, Y has no boundary, and $Z \subset Y$ is a manifold without boundary, if f and ∂f are both transversal to Z, then $f^{-1}(Z)$ is a manifold with boundary $\partial f^{-1}(Z) = f^{-1}(Z) \cap \partial X$. We also have the condition that $f^{-1}(Z)$ has the same codimension as Z in Y.

Fortunately, the idea of transversality can be flushed out even further for manifolds with boundary. In the previous chapter, we showed that transversality is a stable condition, and that furthermore, we can "perturb" a function $f : X \to Y$ ever so slightly to be transversal to some $Z \subset Y$. In the case of manifolds with boundaries, we have something similar. We start off with the transversality theorem, which tells us that given $F : X \times S \to Y$, where only X has boundary, and given $Z \subset Y$ without boundary, if F and ∂F are both transversal to Z, then for almost every $s \in S$, the function f_s (defined on X, taking $x \mapsto F(x, s)$) and ∂f_s are transversal to Z. Again, we have what we want, as well as a mention for ∂f_s, which makes sense, as ∂X is a manifold without boundary. This theorem is rather powerful, in that we can use it to show maps to Euclidean space can be made to be "perturbed" to be transversal to any given submanifold. Given $f : X \to \mathbb{R}^m$, we can define $F : X \times S \to \mathbb{R}^m$, for S being the unit ball in $\mathbb{R}^m$, defined by $(x, s) \mapsto f(x) + s$. By our theorem, we see that given $Z \subset \mathbb{R}^m$, f can be "perturbed" by any arbitrarily small s, to obtain $f_s \pitchfork Z$.

But what if $\mathbb{R}^m$ was replaced with some arbitrary boundaryless Y? Since Y sits in an ambient euclidean space, say $\mathbb{R}^m$, we need to somehow project to Y, given a perturbation on f. This is where the ϵ-neighborhood theorem comes in. Given $Y \subset \mathbb{R}^m$ without boundary, and a positive function $\epsilon : Y \to \mathbb{R}_{>0}$, we define $Y^\epsilon = \{w \in \mathbb{R}^m :$ there is some $y \in Y$ such that $|w - y| < \epsilon(y)\}$. The ϵ-neighborhood theorem tells us that given ϵ to be small enough (ie. $|\epsilon(x)|$ for each x is small enough), the map $\pi : Y^\epsilon \to Y$ taking $w \in Y^\epsilon$ to the unique closest $y \in Y$ is a submersion. We note that when Y is compact, we see that ϵ can be taken to be a constant.

The ϵ-neighborhood theorem can be further used to help show the Transversality Homotopy Theorem, which states that given $f : X \to Y$, and $Z \subset Y$, where only X may have boundary, there is a smooth map $g : X \to Y$ homotopic to f, where g and ∂g are both transversal to Z. It turns out that this theorem can be strengthened by the extension theorem if f and ∂f are transversal to Z in a closed set $C \subset X$ (that is, we are only looking for the transversality condition at the points in $C \cap f^{-1}(Z)$). Suppose we again, have $f : X \to Y$, where $Z \subset Y$, and only X has boundary. Let $C \subset X$ be a closed subset of X. Suppose we have $f \pitchfork Z$ on C and $\partial f \pitchfork Z$ on $C \cap \partial X$. Then, there's a smooth map g homotopic to f where g and ∂g are transversal to Z, and $g = f$ in a neighborhood of C. Since ∂X is closed in X, a special case is when $C = \partial X$. This is rather significant in the case of manifolds with boundaries, in if we only have transversality of f at the boundary of X, we can "redefine" how it is in the interior (leaving how it's defined at the boundary as is) in order to get transversality in all of X.

We now talk about intersection theory mod 2. Let $X, Z \subset Y$. We say X and Z have complementary dimensions if $\dim(X) + \dim(Z) = \dim(Y)$. If $X \pitchfork Z$, then we have that $X \cap Z$ is a 0-manifold.

Moreover, if both are closed and one is compact, then $X \cap Z$ is a finite set that can be counted. We call this the intersection number. Unfortunately, the intersection number is not invariant under homotopy, but the parity of the intersection number is. We call this intersection theory mod 2, which gives either 1 or 0. At first glance, this seems like a useless concept, in that there are only two possible outputs. However, it turns out that there's quite a bit of mileage that we can get out from this idea.

Suppose $f : X \to Y$, where X is compact, $Z \subset Y$ is closed, and $\dim(X) + \dim(Z) = \dim(Y)$.We define $I_2(f, Z)$ to be the number of points in $f^{-1}(Z) \mod 2$. What's nice about intersection mod 2 is that if g is homotopic to f, then $I_2(f, Z) = I_2(g, Z)$. This allows us to define $I_2(f, Z)$ in general, even if $f \not\pitchfork Z$, since we can find some g homotopic to f where $g \pitchfork Z$. In fact, we also see from here that given two homotopic smooth functions g, h, $I(g, Z) = I(f, Z)$. Given $X, Z \subset Y$, we also define $f(X, Z) = f(i, Z)$, where $i : X \hookrightarrow Y$.

One of the results from intersection mod 2 is the Boundary Theorem. The boundary theorem states that if a manifold X is the boundary of a manifold W, and $g : X \to Y$ is a smooth map, if g can be extended to all of W, then $I_2(g, Z) = 0$. This tells us something rather interesting, that maps restricted to the boundary give us an intersection number of 0.

In fact, we can use intersection mod 2 to define an invariant on maps of manifolds of the same dimension. Suppose $f : X \to Y$ where $\dim(X) = \dim(Y)$. We define $\deg(f) = I_2(f, \{y\})$, for some given $y \in Y$. It turns out that the value for this is the same for any $y \in Y$, which means the degree of f is well defined. From this, we can port over a few results from intersection mod 2 to the degree of a function. If f and g are homotopic, then $\deg_2(f) = \deg_2(g)$. We also have that if we have a map $f : X \to Y$ where $X = \partial W$ for some manifold W, and if f can be extended to W, then $\deg_2(f) = 0$.

One of the things we get from the degree mod 2 is the following. Suppose $p : \mathbb{C} \to \mathbb{C}$ and $W \subset \mathbb{C}$ is compact. If p has no zeros on ∂W, we can define $p/|p| : \partial W \to S^1$. We note that if p has no zeros in W, then $p/|p|$ can be extended, which means $\deg(p/|p|) = 0$. By contrapositive, this means that if $\deg(|p|/p) = 1$, then we necessarily have that p has a zero in W. We can use this to show "half" of the Fundamental Theorem of Algebra, that complex polynomials of odd degree have roots in $\mathbb{C}$. This result shows both the potential and limitations of degree mod 2.

We will now use degree mod 2 to define the winding number mod 2 of a function. Let $f : X \to \mathbb{R}^n$, where $\dim(X) = n - 1$. Let $u(x) = \frac{f(x)-z}{|f(x)-z|}$, where z is a point in $\mathbb{R}^n$ not in $f(X)$. We define the winding number mod 2 around z to be $W_2(f, z) = \deg_2(u)$. We can now use the winding number mod 2 to show the Jordan-Brouwer Separation Theorem, which tells us that the complement of a compact connected hypersurface (ie. codimension 1) X in $\mathbb{R}^n$ consists of two connected pen sets, which we call the inside and the outside, and that the closure of the inside gives a compact manifold with boundary being X.

Another place where the winding number mod 2 pops up is the Borsuk-Ulam Theorem. That is, we can show inductively that if we have a map $f : S^k \to \mathbb{R}^{k+1}$ whose image does not contain the origin, and that $f(x) = -f(x)$ for all $x \in S^k$, then $W_2(f, 0) = 1$. This theorem has a number of consequences. For example, if $f : S^k \to \mathbb{R}^{k+1} \setminus \{0\}$ satisfies $f(-x) = -f(x)$, then f intersects every line through the origin at least once. These results show us that although intersection theory and degree mod 2 has its limitations, there are still a lot of consequences derived from it.

Chapters 3
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Guillemin & Pollack

Name: Jiajie (Jerry) Luo

In the previous chapter, we discussed intersection theory mod 2, and showed a number of results derived from it, as well as its certain limitations. This gives us motivation to bring up a different way to look at intersection - through the lens of orientation. To do this, we first introduce a new an orientation on a manifold. That is, we assign a "proper direction" onto our manifold. We do this by looking at the tangent space $T_x(X)$ for some $x \in X$ that varies smoothly.

We start by first looking at finite dimensional vector spaces over $\mathbb{R}$. Let $\beta = \{b_1, \cdots, b_n\}$ be an ordered basis for a vector space V. If we have another basis $\beta' = \{b_1', \cdots, b_n'\}$, we say that β and β' are equivalently oriented if there is an isomorphism A such that $A\beta = \{Ab_1, \cdots, Ab_n\} = \beta'$, where A has positive determinant. So, given a choice of orientation (ie. we take a basis and declare it to be the right orientation), we can now say whether a basis β is correctly oriented (ie. positively oriented) or not (ie. negatively oriented). We notice that if we have an isomorphism $A : V \to W$, if β and β' are the same orientation for V, then $A\beta$ and $A\beta'$ are of the same orientation for W. So, we say that A is orientation preserving if the sign on β and $A\beta$ is the same. Otherwise, A is orientation reversing. We note that this definition does not include $\{0\}$, as it has empty basis, so for this case, we give it a choice of sign ± 1 associated with it.

We now have the necessary tools to look at orientation of a manifold with boundary X. An orientation of X is a smooth choice of orientations for vector spaces. That is, for every $x \in X$, there must be a local parameterization $h : U \to X$ where h is an orientation preserving map. That is, $dh_u : \mathbb{R}^k \to T_{h(u)}(X)$ preserves orientation on every $u \in U$. We note however that not every manifold with boundary has an orientation (consider the Möbius strip). If X is orientable, then X can be assigned two orientations (if we have one orientation, we can simply reverse the orientation on every tangent space to get the other one). It turns out that there are only two orientations on X, if it is connected and orientable. So, given an oriented manifold X, we refer to the other orientation as $-X$.

Now, let X and Y both be oriented manifold, with at least one of them without boundary. A natural question that arises is to determine whether the orientation on X and Y give us an orientation on $X \times Y$. The answer is that it does. We see that the tangent space at (x, y) is $T_{(x,y)}(X \times Y) = T_x(X) \times T_y(Y)$. So, given a basis $\alpha = \{v_1, \cdots, v_n\}$ for $T_x(X)$ and $\beta = \{b_1, \cdots, b_m\}$ for $T_y(Y)$, we construct a basis $(\alpha \times 0, 0 \times \beta) = \{(v_1, 0), \cdots, (v_n, 0), (0, w_1), \cdots, (0, w_m)\}$. We use the orientations of α and β to induce an orientation on $(\alpha \times 0, 0 \times \beta)$ in the following manner: $sign((\alpha \times 0, 0 \times \beta)) = sign(\alpha)sign(\beta)$. From this, we're able to determine the correct orientation on $X \times Y$. In the case of direct sums, say $V = V_1 \oplus V_2$, if α is a basis for V_1 and β a basis for V_2, then the orientation of (α, β), we demand that $sign(\alpha, \beta) = sign(\alpha)sign(\beta)$ (note the basis (β, α) may be differently oriented than (α, β)).

It turns out that given an oriented manifold with boundary X, the orientation on X also induces an orientation on ∂X. We recall that ∂X are the points x where the parameterization about x is a neighborhood of the boundary of H^k. Furthermore, ∂X is a manifold without boundary with codimension 1 in X. We see that in $T_x(X)$, where $x \in \partial X$, there are two vectors, one pointing inwards and one pointing outwards, that are perpendicular to $T_x(\partial X)$. We will use n_x, the outward pointing vector. So, given a basis $\beta = \{v_1, \cdots, v_{k-1}\}$ on $T_x(\partial X)$, we determine the orientation of β by constructing $\beta' = \{n_x, v_1, \cdots, v_{k-1}\}$ and seeing its orientation in $T_x(X)$ (we get the other orientation if we use the inwards pointing vector instead). If β' is properly oriented in $T_x(X)$, then β is oriented, and vice versa.

A special case of this is if we take X, a manifold without boundary, and consider $X \times I$. In this case, we see that $\partial(X \times I) = X \times \{0\} \cup X \times \{1\}$, which we denote as $X_0 \cup X_1$. With a given orientation on $X \times I$, we can use what we did above to induce orientations on X_0 and X_1, both of which are copies of X. We see that if we take a point $x \in X$ (this is an abuse of notation - we are looking at a point on X_0 and its corresponding one on X_1), the inward pointing perpendicular vector to $T_x(X_0)$ is the direction of the outward pointing perpendicular vector to $T_x(X_1)$, which comes to show that the orientation on X_0 and X_1 are actually reversed. A corollary of this is that the sum of orientation numbers at the boundary points of compact oriented 1-dimensional manifolds with boundary is 0, because each path component is diffeomorphic to S^1 (which has no boundary) or $[0, 1]$ (where the ends have opposite orientations, thus canceling out).

Now, suppose $f : X \to Y$ is a smooth map where $Z \subset Y$ is a submanifold with $f \pitchfork Z$ and $\partial f \pitchfork Z$, and Z and Y without boundary. Can we say anything about the orientation induced on $f^{-1}(Z)$? The answer is yes, which we define as the preimage orientation on $S = f^{-1}(Z)$. Given $x \in f^{-1}(z)$, for $z \in Z$, we have that $T_x(S)$ is the preimage of $T_z(Z)$ under $df_x : T_x(X) \to T_z(Z)$. We obtain from this the direct sum $N_x(S; X) \oplus T_x(S) = T_x(X)$, where $N_x(S; X)$ is the orthogonal complement of $T_x(S)$ in $T_x(X)$. Since we also have $df_x(T_x(X)) + T_z(Z) = T_z(Y)$ by transversality, we also obtain the following direct sum: $df_x(N_x(S; X)) \oplus T_z(Z) = T_z(Y)$. So, we see that the orientation of Z and Y at z give us an orienation on $df_x(N_x(S; X))$, and using the first direct sum equation, we additionally use the orientation on X at x to give us an orientation on S.

We've just shown two induced orientations: one for a manifold with boundary on its boundary, and the other with taking the preimage of a submanifold given transversality. A special case involves the use of both of these. Suppose $f : X \to Y$, where $Z \subset Y$, and Y and Z are boundaryless. We're also given that $f \pitchfork Z$ and $\partial f \pitchfork Z$, with orientations on all of our objects. We now have two ways to look at the orientation of the boundary of $f^{-1}(Z)$: as the preimage of Z under ∂f, and as the boundary of $f^{-1}(Z)$. It turns out that the induced orientations are not always the same, and that they are related by the following relation: $\partial[f^{-1}](Z)] = (-1)^{codim(Z)}(\partial f)^{-1}(Z)$. This is a bit weird, in that in this case, the two perspectives that we have in looking at the pullback of a submanifold, which we want (or should I say, I want) to be the same, offer two different stories.

We now have the tools to develop oriented intersection theory. We start off with the same set up as the mod 2 in the previous chapter, but now, all our objects are oriented. That is, given $f : X \to Y$, X, Y, Z are all without boundary, X is compact, Z is a closed manifold of Y, and X and Z have complementary dimension in Y. If $f \pitchfork Z$, then $f^{-1}(Z)$ is a finite number of points, with orientation ± 1 provided by the preimage orientation. We define $I(f, Z)$, the orientation intersection number, to be the sum of these orientation numbers. At any $x \in f^{-1}(Z)$, since we have complementary dimensions and transversality, we have that $df_x(T_x(X)) \oplus T_z(Z) = T_z(Y)$, for $f(x) = z$.

Oriented intersection theory has some parallel properties with intersection theory mod 2. If $X = \partial W$ where W is compact and $f : X \to Y$ extends to W, then we have that $I(f, Z) = 0$. Moreover, we also have that homotopic maps have the same intersection number. This also gives rise for us to define $I(f, Z)$ where f is not transversal, since we can just find a homotopy equivalent map that is.

With the intersection number, it makes sense to define the degree of a function. With the exact same set up as the mod 2 case (recall that $\dim(X) = \dim(Y)$), we define the degree of a function the same way we did so in the mod 2 case: $\deg(f) = I(f, \{y\})$. We can show similarly that this value does not depend on the choice of y. Now, what we have is that this value counts the preimage points, with each contributing a $+1$ or -1, depending on whether $df_x : T_x(X) \to T_y(Y)$ is orientation preserving or reversing. As before, we have some similar properties as that of degree mod 2, such as the following: If $f : X \to Y$ is a smooth map of complex oriented manifolds, where $X = \partial W$ for some W, then $\deg(f) = 0$. We also can use oriented intersection theory to show the Fundamental Theorem of Algebra (which we recall, intersection theory mod 2 failed to do), further demonstrating its potency.

What if we have two submanifolds $X, Z \subset Y$? Recall in intersection theory mod 2, when $X \pitchfork Z$, we had the inclusion map $i : X \hookrightarrow Y$ and we looked at $I(X, Z) = I(i, Z)$ (ie. count the intersection and take mod 2). In the case for oriented intersections, we simply do the same thing. However, we note that in this situation, order matters, which means $I(X, Z) \neq I(Z, X)$ in general. In fact, when X and Z are both compact, we actually have that $I(X, Z) = (-1)^{\dim(X)\dim(Z)} I(Z, X)$.

We now look at $f : X \to Y$ and $g : Z \to Y$, where f and g are injective maps and $\dim(X) + \dim(Z) = \dim(Y)$. Here, we say that $f \pitchfork g$ if $df_x T_x(X) + dg_z T_z(Z) = T_y(Y)$, where $f(x) = g(z) = y$. In this case, we actually have $df_x T_x(X) \oplus dg_z T_z(Z) = T_y(Y)$, due to complementary dimensions. So, it makes sense for us to define $I(f, g)$. To do this, suppose we have $y \in Y$ such that $f(x) = g(z) = y$. In this case, we assign give y a positive orientation (ie. $+1$) if the induced orientation of $df_x T_x(X) \oplus dg_z(Z)$ is the correct orientation of $T_y(Y)$. Otherwise, we give y a negative orientation (ie. -1). $I(f, g)$ is defined to be the sum of all of the ± 1's that we obtain from the y's in the images of both f and g. Using arguments on ordered bases, we're able to show that $I(f, g) = (-1)^{dim(X)dim(Z)} I(g, f)$. One interesting thing about this is that $I(f_0, g_0) = I(f_1, g_1)$ when $f_0 \simeq f_1$ and $g_0 \simeq g_1$, which tells us that I is invariant under homotopies. Using these tools, we develop a new invariant for a compact oriented manifold Y: the Euler characteristic. We say that the Euler characteristic is defined to be $\chi(Y) = I(\Delta_Y, \Delta_Y)$, where Δ_Y is the diagonal of Y, living in $Y \times Y$. Here, we see that in order for this to be defined, we need to "perturb" one of the Δ_Y's, in order to obtain transversality. One thing we note from the results above is that if Y is odd dimensional, then so is Δ_Y, from which, we deduce that $\chi(Y) = -\chi(Y)$, which means $\chi(Y) = 0$.

We can actually take the notion of Euler characteristic and generalize it further. Recall that if $f : X \to Y$, $graph(f) = \{(x, f(x)) : x \in X\}$. Given $f : X \to X$, where X is compact, oriented, and without boundary, we define the Lefschetz number $L(f) = I(graph(f), \Delta_X)$. We notice that $\chi(X) = L(id_X)$. One of the things we can draw from the Lefschetz number is the Smooth Lefschez Fixed-point theorem, which tells us that given $f : X \to X$ where X is compact and oriented, if $L(f) \neq 0$, then f has a fixed point. This theorem tells us quite a bit about the rigidity of compact oriented manifolds, and is, in a way, a generalization of the Brouwer fixed point theorem. It also makes quite a bit of sense, as the Lefschetz number counts how many times Δ_X and $graph(f)$ intersect, which counts the fixed points. We can further show that the Lefschetz number is a homotopy invariant.

We note that from above, we may have that $L(f)$ may not even be defined, because there could be an infinite intersection. However, we are able to remedy this. We first introduce a class of maps called Lefschetz maps, where $Graph(f) \pitchfork \Delta_X$. In this case, since we also have complementary dimensions, $L(f)$ is certainly defined, since the intersection is finite. It turns out that every map $f : X \to X$ is homotopic to a Lefschetz map. Since $L(f)$ is invariant under homotopy, the definition of $L(f)$ is well defined, as $L(f) = L(g)$, where $f \simeq g$ and g is a Lefschetz map.

Let x be a fixed point of f. We say that x is a Lefschetz fixed point of f if df_x has no nonzero points.

This is equivalent to 1 not being an eigenvalue of df_x, meaning $df_x - I$ is invertible. We're also to see that f is a Lefschetz map if and only of all its fixed points are Lefschetz. Now, let us define the local Lefschetz number at a given fixed point x. We say that $L_x(f)$ is ± 1, depending on the orientation of (x, x) in $graph(f) \cap \Delta_X$. We see that for Lefschetz maps, we have that $L(f) = \sum_{x=f(x)} L_x(f)$. In fact, we can show that at a Lefschetz fixed point x, $L_x(f) = 1$ if $df_x - I$ preserves orientation, and $L_x(f) = -1$ if $df_x - I$ reverses orientation.

In order to give a sense of how the local Lefschetz number can be used, let us look at how we can use it to determine local properties of certain 2-manifolds. Since it's the local properties we are concerned about, let us consider a map $f : \mathbb{R}^2 \to \mathbb{R}^2$ fixing the origin, and let $A = df_0$. If A has two linearly independent real eigenvectors, we can let A be a diagonal matrix under some basis. let us assume its diagonal (ie. its nonzero) entries (call them α_1, α_2) are positive. In this situation, we see that if $\alpha_1, \alpha_2 > 1$ or $\alpha_1, \alpha_2 < 1$, we have that $L_0(f) = 1$, with the former being an "expanding map" and the latter being a "contracting map". If we have $\alpha_1 < 1 < \alpha_2$ or $\alpha_2 < 1 < \alpha_1$, we have that $L_0(f) = -1$, and 0 a saddlepoint of f. We can use this fact to tell us the Euler characteristic of genus k surfaces. To do this, given a genus k surface X, we can construct a map $g : X \to X$ with fixed points such that we have one source and one sink, and each hole has two saddle points (see picture below). As this is invariant under homotopy, we see that our genus k surface has Euler characteristic $2 - 2k$.

We ended the course with the Poincaré-Hopf Theorem, which connects vector fields to the Euler characteristic. Recall that a vector field on a manifold X embedded in $\mathbb{R}^N$ is a smooth assignment of vectors $\vec{v} : X \to \mathbb{R}^N$ such that $\vec{v}(x) \in T_x(X)$ for all x. For an isolated zero x, we say that the index of x as $\deg(\vec{v}/|\vec{v}|)$, where $\vec{v}/|\vec{v}| : S_\epsilon \to S^k$ (where S_ϵ is an ϵ-sphere around x such that x is the only fixed point in or on S_ϵ) is defined by $x \mapsto \vec{v}(x)/|\vec{v}(x)|$. With this defined, the Poincaré-Hopf Theorem tells us that if $\vec{v}$ is a smooth vector field on a compact oriented manifold X with finitely many zeros, then the sum of the indices around the zeros is equal to the Euler characteristic of X (ie. $\sum\limits_{x:\vec{v}(x)=0} ind_x(\vec{v}) = \chi(X)$). This fact connects vector fields to the Euler characteristic, and allows us to use a vector field over a compact oriented manifold to determine its Euler characteristic.